Intrinsic Mirror Symmetry

June 22^{nd}, 2021

Mark Gross (University of Cambridge)

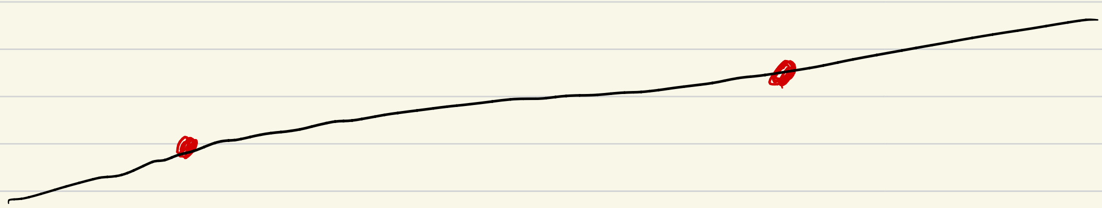
joint work with

Bernd Siebert (UT Austin)

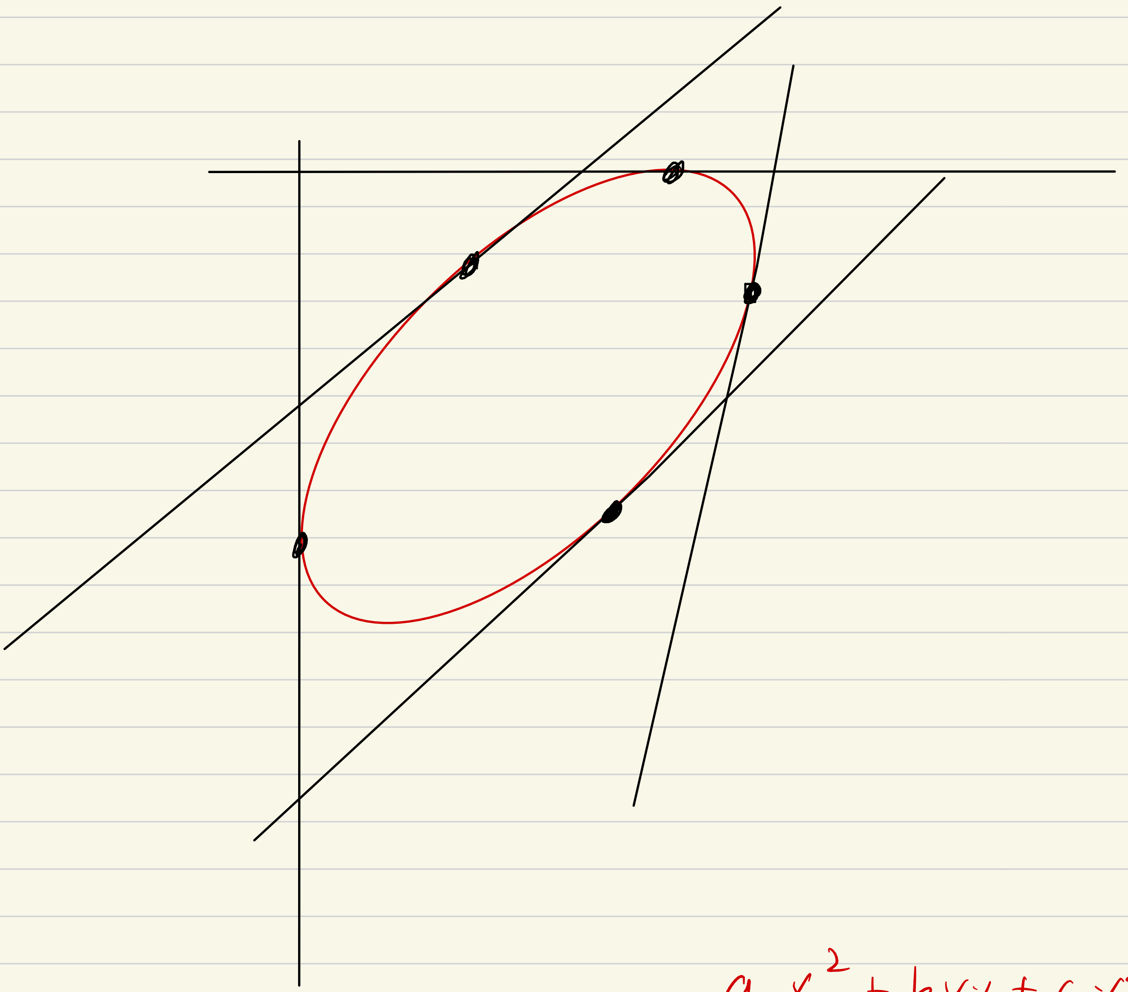
Enumerative geometry (~ 19^{th} century)

We ask questions such as:

① How many lines pass through 2 points in the plane?



② Given 5 lines in the plane, how many conics are tangent to all 5 lines?



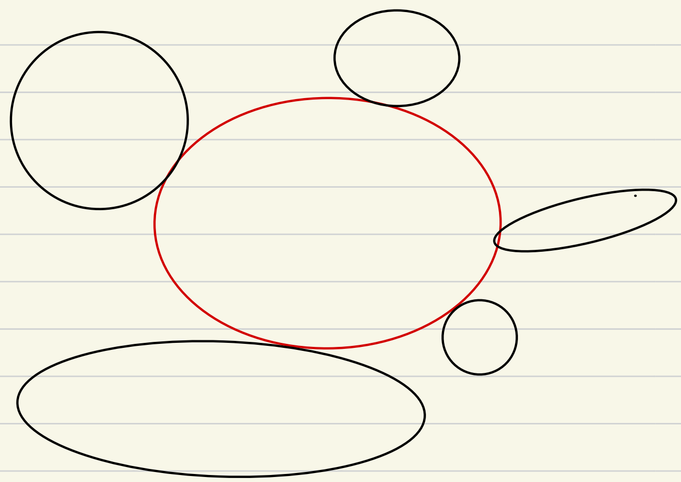
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Being tangent to a given line is a quadratic condition on coefficients $a, \dots, f$.

Answer: 1

(3) How many conics are tangent to 5 fixed conics in the plane?

(Steiner's conic problem, 1848)



Answer!

3,264

Caution: There are three issues one needs to deal with to get this number.

1. Instead of working with conics with real coefficients, we should count conics with complex coefficients

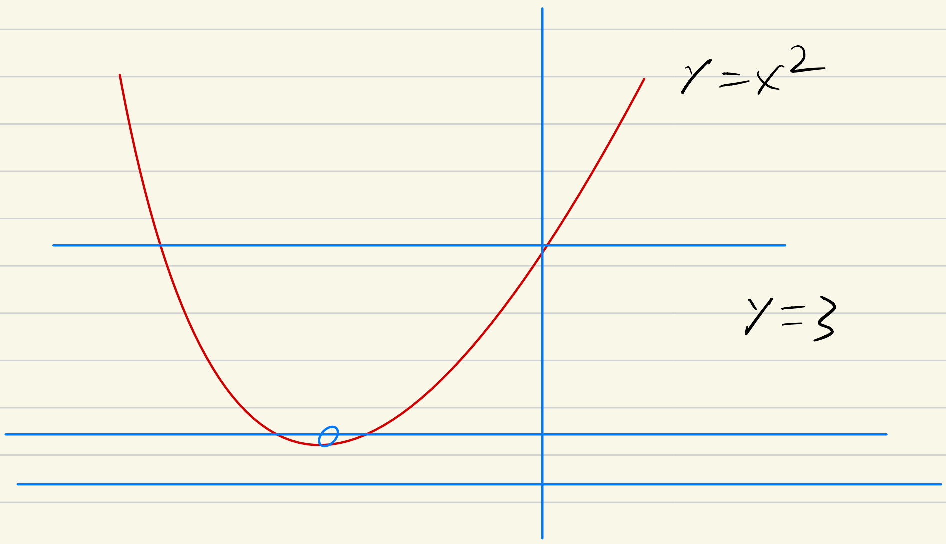
[A polynomial of degree d will have d complex solutions, counted with multiplicity, but not necessarily d real solutions.]

2. We should understand how to "count multiplicities."

Subtle!

③ We should make sure solutions to equations don't "escape to infinity"

e.g.



To deal with this, we work in complex projective space

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\} \right) / \sim$$

$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ for $\lambda \in \mathbb{C} \setminus \{0\}$.

If $f(x_0, \dots, x_n)$ is a homogeneous polynomial of degree d, then

$$Z(f) = \{ (a_0, \dots, a_n) \in \mathbb{CP}^n \mid f(a_0, \dots, a_n) = 0 \}$$

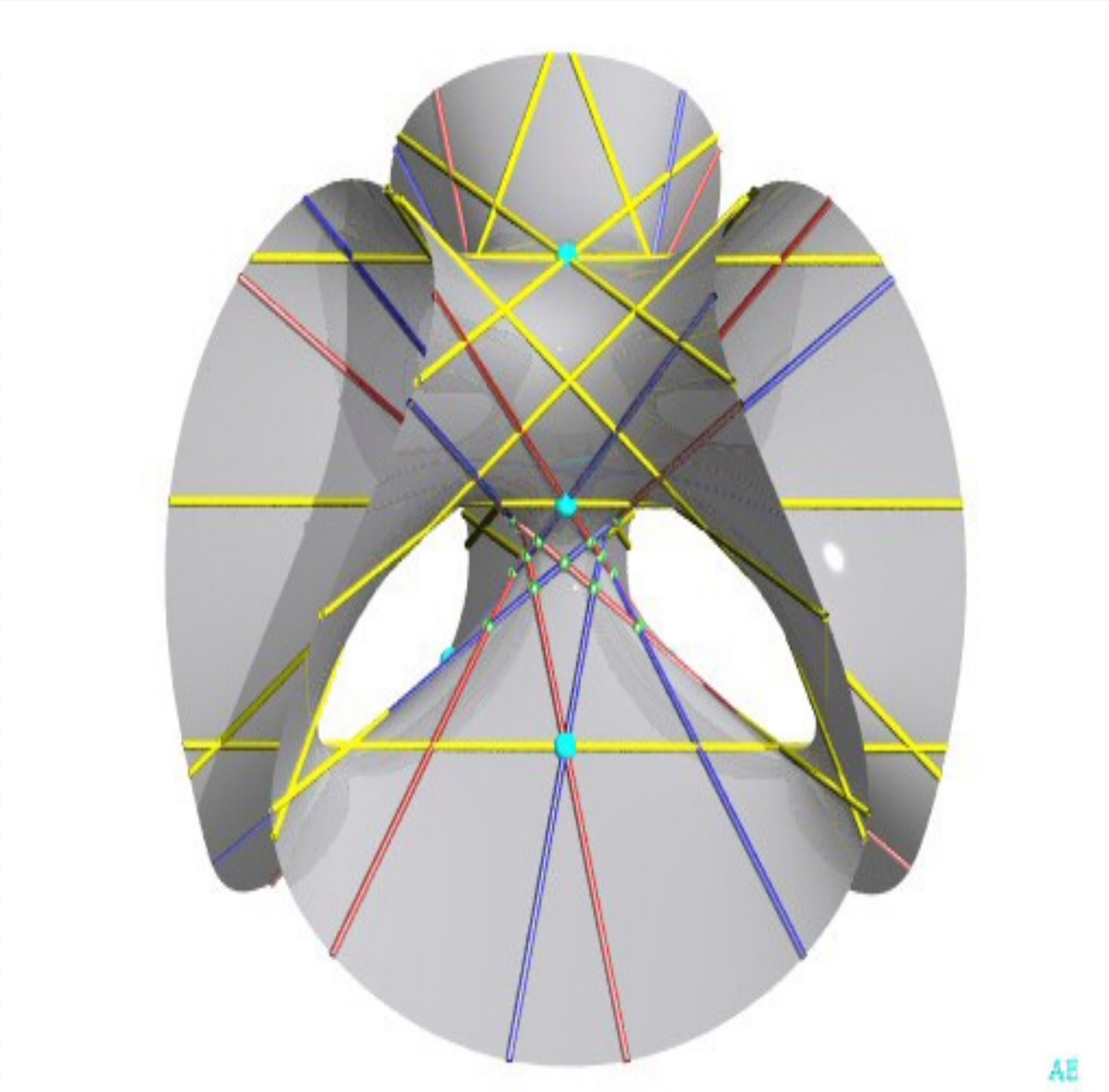
makes sense and is a <u>hyper-surface of degree d</u>.

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$$

<u>Example</u>: (Clebsch diagonal cubic surface)

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3$$

$$\subseteq \mathbb{CP}^3$$

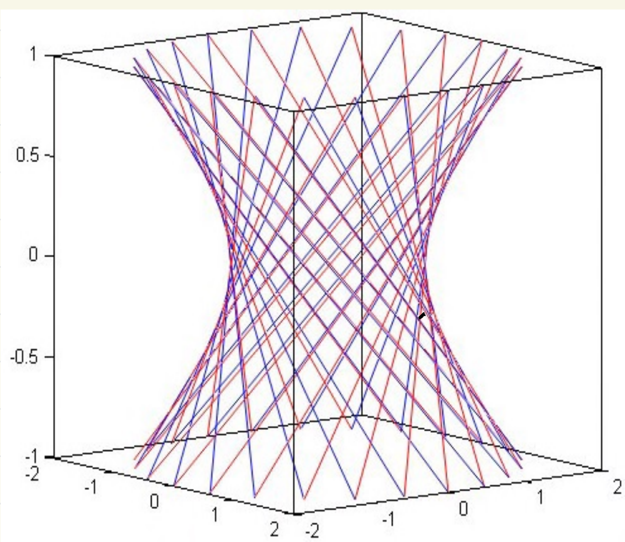

AE

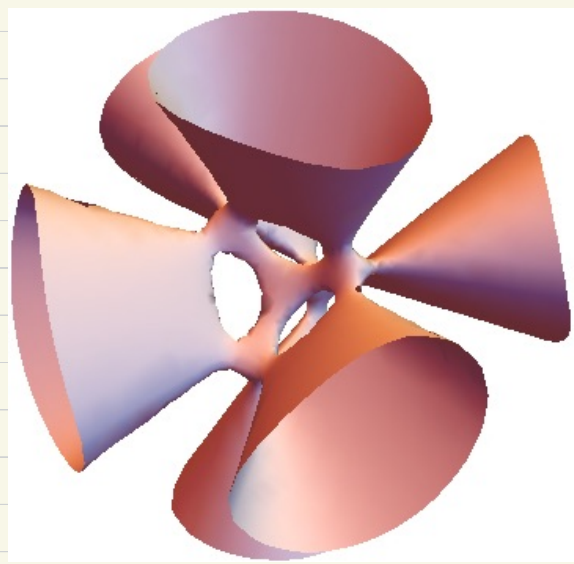
Every cubic surface which is a manifold Lie, is <u>non-singular</u> & contains precisely 27 lines.

(Cayley-Salmon, 1849)

(Arthur Cayley's final resting place)

Question: Degree 3 surfaces are special, as degree ≤ 2 surfaces will contain an infinite number of lines





What about hyper-surfaces in $\mathbb{CP}^4$?

The critical degree is 5.

(H. Schubert, 19th century)

The general non-singular quintic 3-fold has 2,875 lines.

Unlike the cubic surface, the general quintic 3-fold will contain a finite number of conics, a number calculated by Sheldon Katz (1986) to be

609,250

What about higher degrees?

Let us consider maps

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^4$$
$$u, v \qquad\qquad x_0, \dots, x_4$$

defined by

$$f(u,v) = (f_0(u,v), \dots, f_4(u,v))$$

where

- $f_0, \dots, f_4$ are homogeneous polynomials of degree d with no common zeros.
- f is injective on a dense open subset of $\mathbb{CP}^1$.

We consider $f, f' : \mathbb{CP}^1 \to \mathbb{CP}^4$

the same if $\exists$ an automorphism

$\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that

$f \circ \varphi = f'$. (Reparametrization)

Fix a non-singular quintic $X \subseteq \mathbb{CP}^4$,

and let N_d = # of such maps of degree d, with image contained in X.

$N_1 = 2{,}875$, $N_2 = 609{,}250$

$N_3 =$ 317,206,375

Ellingsrud + Strømme, 1990.

Enter string theory, 1990.

Candelas, de la Ossa, Green and Parkes

Used string theoretic methods to find a conjectural generating series for the sequence N_d, $d \geq 1$.

Key point: String theory predicts the existence of a mirror manifold to X, i.e., a variety $\check{X}$ such that

$$\chi_{top}(X) = -\chi_{top}(\check{X}).$$

The generating function for N_d

then can be calculated via

period integrals on $\check{X}$:

$$\int_\alpha \Omega$$

Ω a nowhere vanishing holomorphic 3-form on $\check{X}$.

α 3-cycle on $\check{X}$

(Existence of such is the Calabi-Yau condition, and both X, $\check{X}$ satisfy this condition.)

Many important questions

① Rigorous definition of the numbers N_d. (Ruan-Tian, Behrend-Fantechi, ...) '92 - '95

$\rightsquigarrow$ Gromov-Witten invariants,

② Proof of physics predictions

(Givental, '96, Lian-Liu-Yau, '97, ...)

③ Construction of mirrors

(Batyrev, '93 - combinatorial construction of $\sim 2 \times 10^8$ mirror pairs)

(constructed mirror pairs which were hypersurfaces in toric varieties.)

(4) Why does mirror symmetry work?

Homological mirror symmetry (Kontsevich, '94)

Symplectic geometry of X $\cong$ algebraic geometry of $\check{X}$

Strominger-Yau-Zaslow conjecture ('96)

Program joint with Bernd Siebert (since 2001)
Algebro-geometric approach motivated by SYZ conjecture.

Mirror pairs are dual "special Lagrangian" torus fibrations

$$\begin{tikzcd} X \arrow[dr] & & \check{X} \arrow[dl] \\ & B & \end{tikzcd}$$

$$\dim_{\mathbb{R}} B = \dim_{\mathbb{C}} X$$

General constructions of mirrors.

"Intrinsic mirror symmetry" G - Siebert, 2019.

(Similar construction in special case, Keel-Yu, 2019.)

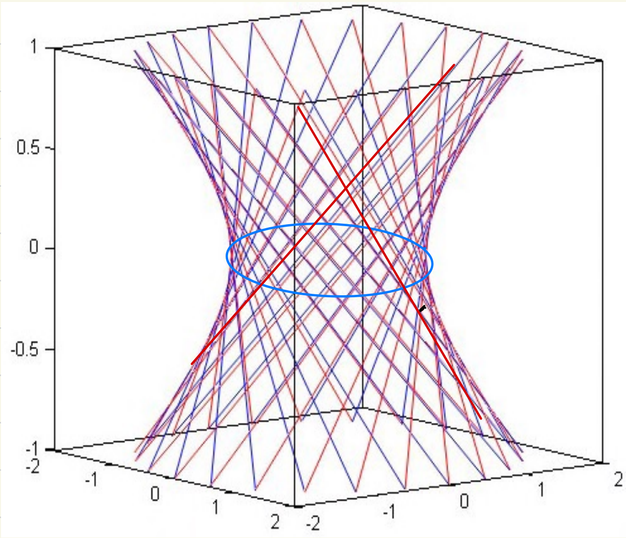
Simplification for this discussion.

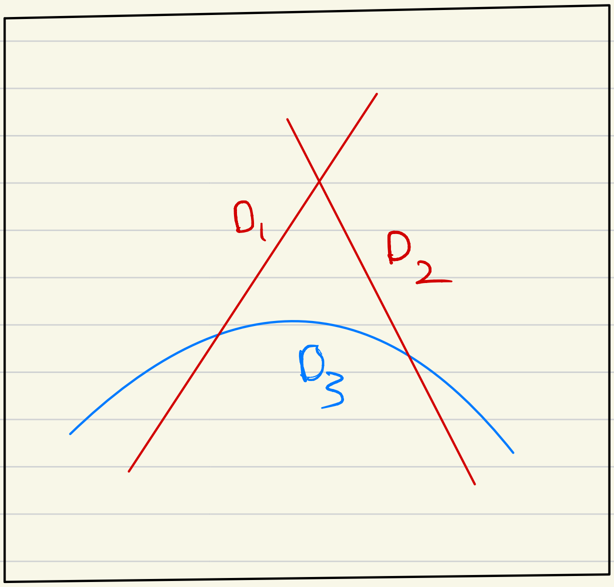
Work with a <u>log Calabi-Yau pair</u>,

i.e., a pair (X, D) with X a non-singular variety, D a union of hypersurfaces in X meeting transversally and such that $X \setminus D$ carries a top rank nowhere vanishing holomorphic form with simple poles along D.

This is the "log Calabi-Yau" condition.

Example: X = Quadric surface.





Goal: Construct ring of algebraic functions of mirror, i.e., the coordinate ring of the mirror.

Consider maps $f : \mathbb{CP}^1 \to X$.

Given a point $x \in \mathbb{CP}^1$, we can define the <u>contact order</u> of f with D_i at x.

- The contact order is 0 if $f(x) \notin D_i$.
- If $f(x) \in D_i$, and $t=0$ is a local equation for D_i at $f(x)$, then the contact order of f at x is the order of vanishing of $t \circ f$ at x.

Logarithmic Gromov-Witten invariants.

(G-Siebert, Abramovich-Chen, 2010)

Count maps

$$f:(C, x_1, \dots, x_n) \longrightarrow (X, D)$$

with imposed contact orders at distinct points $x_1, \dots, x_n \in C$.

We can record contact orders via formal linear combinations

$$p = \sum a_i D_i, \quad a_i \geq 0, \text{ indicating}$$

contact order a_i with D_i.

$f(x) \in D_i$ $\forall i$ s.t. $a_i > 0$, so $\bigcap_{i \text{ s.t. } a_i > 0} D_i \neq \emptyset$

Note! In order for p to be useful, we should have

$$\bigcap_{i : a_i > 0} D_i \neq \emptyset,$$

in which case we will call p *effective*.

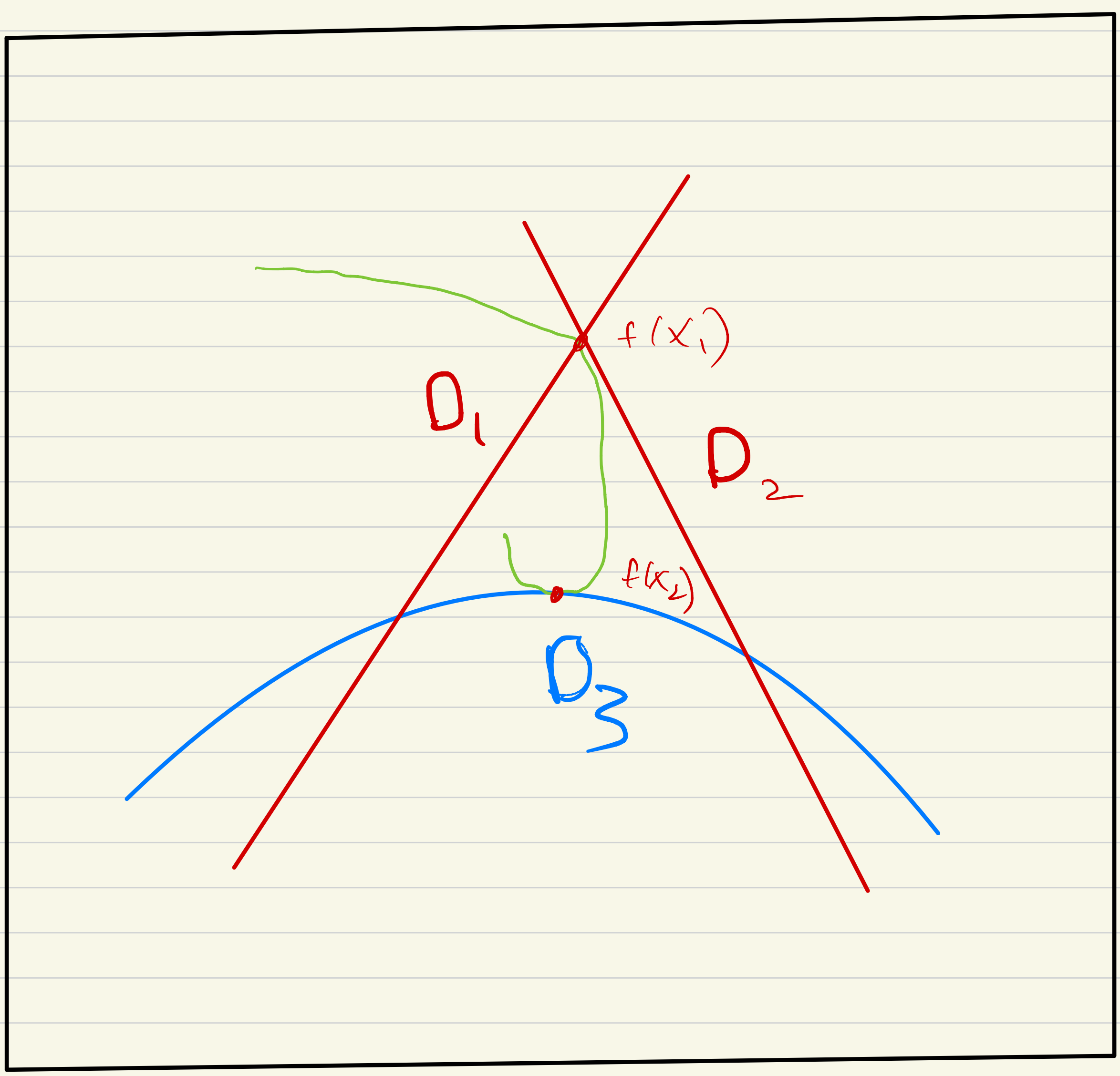
Theorem (AC, GS) $\exists$ a good counting theory for such maps, called

logarithmic Gromov-Witten invariants

E.g. $(C, x_1, x_2) \longrightarrow (X, D)$

Contact orders $p_1 = D_1 + 2D_2$

$p_2 = 3D_3$



The coordinate ring of the mirror:

(first approximation)

Let A be the $\mathbb{C}$-algebra with basis

\vartheta (theta functions)

$$\{\theta_p \mid p \text{ an effective contact order}\}.$$

$$\theta_p \cdot \theta_q = \sum_r N_r \theta_r$$

$N_r = \#$ maps $f: (\mathbb{C}\mathbb{P}^1, x_1, x_2, x_{out}) \longrightarrow (X, D)$

satisfying some constraints.

If $r = \sum a_i D_i$, pick a point

$z \in \bigcap_{i : a_i > 0} D_i \neq \emptyset$ since r is effective.

We require:

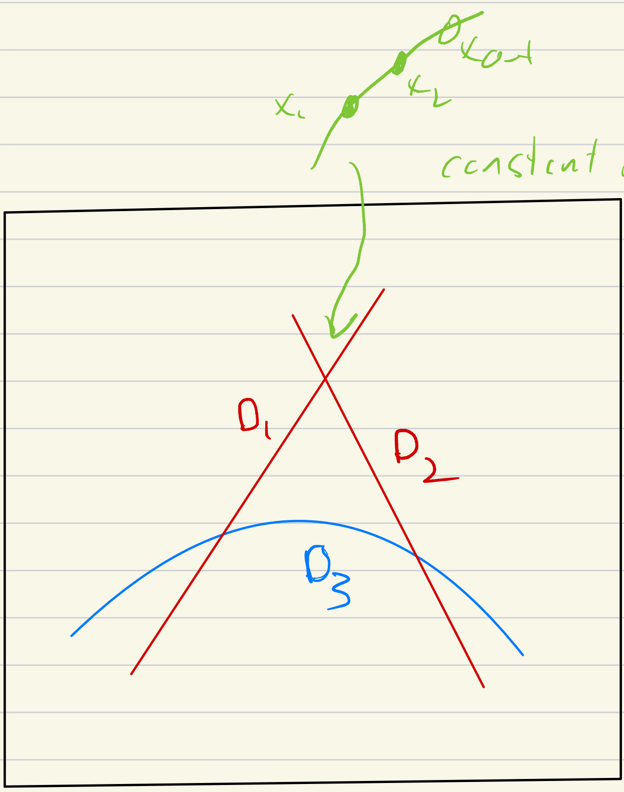
- Contact order at x_1 is p.
- Contact order at x_2 is q.
- $f(x_{out}) = z$ and contact order at x_{out} is $-r$.

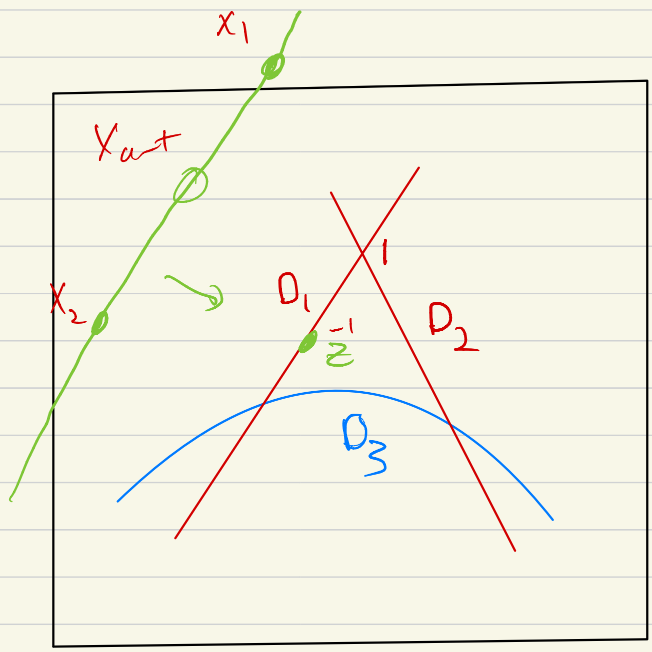
Abramovich-Chen-G.-Siebert ('16, '19, '20) introduce <u>punctured log Gromov-Witten invariants</u> in order to interpret negative contact order!

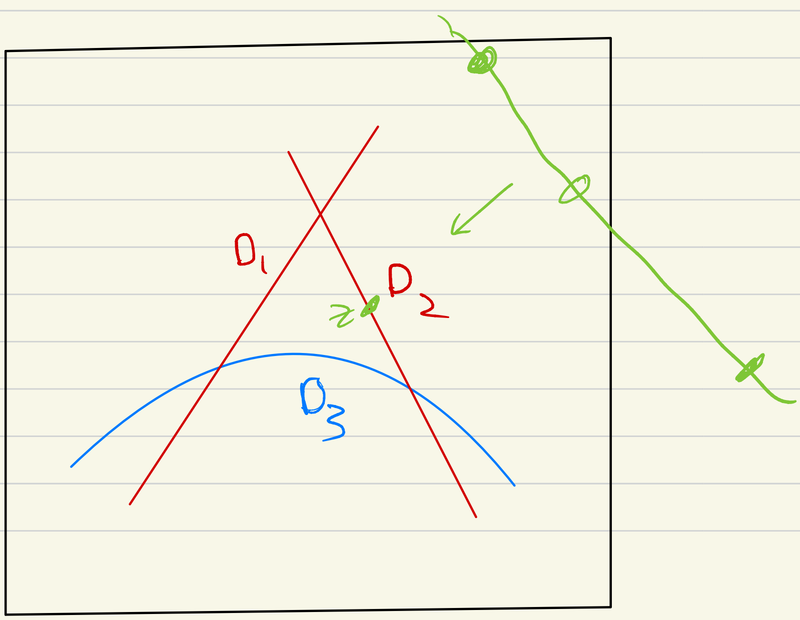
Example:

$$\theta_{D_1}\cdot\theta_{D_2} = \theta_{D_1+D_2}$$

$$\theta_{D_1}\cdot\theta_{D_2}\cdot\theta_{D_3} = \theta_{D_1+D_2}\cdot\theta_{D_3} = \theta_{D_1}+\theta_{D_2}$$







Mirror coordinate ring:

$$\frac{\mathbb{C}[\theta_1, \theta_2, \theta_3]}{(\theta_1\theta_2\theta_3 - \theta_1 - \theta_2)}$$

$\theta_{p_1}, \dots, \theta_{p_3}$

<u>Theorem:</u> (G-S '19) The above construction of a product (after suitably taking into account convergence issues) yields a commutative, associative $\mathbb{C}$-algebra with unit.

Hard!

θ_0

Thank You!

Calabi-Yau case:

$\mathcal{X}_0 \subseteq \mathcal{X}$ ← proper

$\downarrow \quad \downarrow$ maximally unipotent degeneration

$0 \in S$ normal crossings.

S non-compact

Apply construction to

the pair $(\mathcal{X}, \mathcal{X}_0)$.

$\rightsquigarrow$ get ring A with a

natural grading.

Mirror is $\mathrm{Proj}\, A$.